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Partially invariant solutions of a class of nonlinear Schrödinger equations

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Abstract. Partially invariant solutions of a general class of nonlinear Schrödinger equations, involving four arbitrary functions of the modulus ρ of the solution and its derivative ρ_x, are obtained. The modulus $\rho(\xi)$ is assumed to depend on a symmetry variable ξ, whereas the phase $\omega(x, t)$ depends on both independent variables. Both ρ and ω are obtained explicitly, as are the conditions on the coefficients in the equation, necessary for such solutions to exist.

1. Introduction

The purpose of this article is to study partially invariant solutions of a rather general class of nonlinear Schrödinger equations (NLSEs), namely

$$\mathrm{i}u_t + u_{xx} = (F + \mathrm{i}K)u + (G + \mathrm{i}L)u_x \tag{1.1}$$

where $u(x, t)$ is a complex function of two real variables and F, K, G and L are real functions of $|u|$ and $|u|_x$.

Partially invariant solutions of systems of partial differential equations (PDEs) were introduced by Ovsiannikov some time ago [1]. A systematic study of such solutions were initiated in a recent article [2], devoted to partially invariant solutions of complex nonlinear Klein–Gordon ($\varepsilon = -1$) or Laplace ($\varepsilon = +1$) equations of the form

$$u_{tt} + \varepsilon u_{xx} = f(|u|)u \qquad \varepsilon = \pm 1. \tag{1.2}$$

The general theory of partially invariant solutions has been outlined for arbitrary systems of PDEs [1, 2]. In particular, a definition was given in [2], distinguishing partially invariant solutions from invariant ones. In order to make this article readable on its own, we give a very brief summary of the relevant concepts.

Let us consider a system of N PDEs

$$\begin{gathered}\Delta^\mu(x, u, u_{x_i}, u_{x_i x_k}, \ldots) = 0 \\ \mu = 1, \ldots, N \qquad x \in R^p \qquad u \in R^q \qquad q \geqslant 2.\end{gathered} \tag{1.3}$$

Let $\mathcal{G}$ be the symmetry group of equation (1.3), i.e. a local Lie group of local point transformations, leaving the equations (1.3) invariant. Let $\mathcal{G}_0 \subseteq \mathcal{G}$ be a subgroup of $\mathcal{G}$.

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The subgroup $\mathcal{G}_0$ acts on the space $X \times U$ of independent and dependent variables and sweeps out certain orbits in this space. The generic orbits are the level sets of a set of invariants

$$I_1(x, u), \ldots, I_K(x, u). \tag{1.4}$$

If we have $K \geqslant q$ and if the Jacobian

$$J = \frac{\partial(I_1 \ldots I_K)}{\partial(u_1 \ldots u_q)} \tag{1.5}$$

has rank q, then the dependent variables u_i can be expressed in terms of q invariants $I_\alpha \equiv F_\alpha$, $\alpha = 1, \ldots, q$. These can then be viewed as functions of the remaining $K - q < p$ invariants $\xi_1, \ldots, \xi_{K-q}$

$$u_i = U_i(x_1, \ldots, x_p, F_1(\xi), \ldots, F_q(\xi)) \qquad i = 1, \ldots, q. \tag{1.6}$$

Substituting u_i back into equation (1.3) we obtain a reduced system of equations for $F_i(\xi)$. The solutions of this system are called 'invariant solutions' and they are invariant under the subgroup $\mathcal{G}_0$.

If the rank condition (1.5) is not satisfied and we have

$$\operatorname{rank} J = q' < q \tag{1.7}$$

we can express only q' variables u_i in terms of invariants. The remaining functions $u_{q'+1}, \ldots, u_q$ depend on all the variables $x_1, \ldots, x_p$. Substituting back into the original system we obtain a mixed system of equations in which $u_1, \ldots, u_{q'}$, depend on fewer variables than the remaining unknowns. The system is in general inconsistent and compatibility conditions must be imposed. If solutions exist that are not invariant under $\mathcal{G}_0$, or some other subgroup $\tilde{\mathcal{G}}_0$ of the symmetry group $\mathcal{G}$, we obtain 'partially invariant solutions'.

As a matter of fact, we reserve this name for solutions that are not invariant under any subgroup of the symmetry group $\mathcal{G}$. To our knowledge, the first examples of such 'genuinely' partially invariant solutions were obtained in [2]. In particular, it was shown that such solutions exist for any function $f(|u|)$ in equation (1.2) with $\varepsilon = -1$, but only for very special functions $f(|u|)$ for $\varepsilon = +1$.

Our aim is to establish conditions on the functions F, K, G and L, under which genuinely partially invariant solutions exist, and then to obtain these solutions. We emphasize that from the physical point of view partially invariant solutions are just as useful as invariant ones. In particular, they can be used to satisfy different types of boundary conditions than the invariant ones.

A sizable literature exists on NLSEs of the form (1.1). In particular Clarkson recently [5] applied a direct method of dimensional reduction, due to Clarkson and Kruskal [6] to a special case of (1.1) with

$$\begin{aligned} F &= -(a_1+b_1)|u|_x^2 - c|u|^4 - d|u|^2 \\ K &= -(a_2+b_2)|u|_x^2 \\ G &= -a_1|u|^2 \\ L &= -a_2|u|^2 \end{aligned} \tag{1.8}$$

where a_1, a_2, b_1, b_2, c and d are real constants.

This special case includes well known integrable equations, such as the cubic NLSE [7] ($a_1=a_2=b_1=b_2=c=0$, $d\neq 0$), and various derivative NLSEs (e.g. $a_1=b_1=b_2=c=d=0$, $a_2\neq 0$) [8]. It also includes equations linearizable by contact transformations, such as the Eckhaus equation ($a_1=a_2=b_2=d=0$, $c=\frac{1}{2}b_1^2$) [9]. The non-integrable quintic NLSE is included for $a_i=b_i=0$, $c\neq 0$. Its symmetries and group invariant solutions have been studied in detail elsewhere [10, 11].

Various equations of the form (1.1) and in particular (1.8) have been derived in the context of nonlinear optics [12–15], nonlinear water waves [16, 17] and other applications ([5] contains an extensive bibliography).

In section 2 we discuss the symmetries of equation (1.1) and determine the subgroups, providing invariant and partially invariant solutions. In the generic case equation (1.1) is only invariant under space and time translations and changes of phase, generated by P_1, P_0 and W respectively.

In section 3 we show that the subgroup corresponding to $\{P_1, W\}$ provides partially invariant solutions for any function F, K and L, but their existence requires

$$G(|u|, |u|_x=0)=0.$$

The subalgebra $\{P_0, W\}$, as shown in section 4, also leads to partially invariant solutions. Their existence imposes constraints on the coefficients of equation (1.1).

2. Symmetries of the equation

Partially invariant solutions exist only for systems of PDEs. Equation (1.1) will be considered as a system of two real PDEs. We put

$$u(x,t)=\rho(x,t)\,e^{i\omega(x,t)} \qquad 0\leqslant\rho, 0\leqslant\omega<2\pi$$

and rewrite (1.1) as

$$\rho_{xx}-\rho\omega_x^2-\rho\omega_t=F\rho+G\rho_x+L\rho\omega_x \tag{2.1a}$$

$$\rho_t+2\rho_x\omega_x+\rho\omega_{xx}=K\rho+G\rho\omega_x+L\rho_x. \tag{2.1b}$$

Equation (1.1) (and the system (2.1)) is invariant for any F, K, G and L, under space and time translations and under the addition of a constant to the phase. The corresponding Lie algebra (symmetry algebra) is the Abelian algebra generated by

$$P_0=\partial_t \qquad P_1=\partial_x \qquad W=\partial_\omega. \tag{2.2}$$

In special cases the symmetry algebra can be larger. For instance, it includes Galilei transformations

$$B=t\partial_x+\tfrac{1}{2}x\partial_\omega \qquad \text{for } G=L=0 \tag{2.3}$$

and dilations

$$D=2t\partial_t+x\partial_x+p\rho\partial_\rho \tag{2.4a}$$

if we have

$$\begin{aligned}
F(\rho\,e^{p\lambda}, \rho_x\,e^{(p-1)\lambda})&=e^{-2\lambda}F(\rho,\rho_x)\\
K(\rho\,e^{p\lambda}, \rho_x\,e^{(p-1)\lambda})&=e^{-2\lambda}K(\rho,\rho_x)\\
G(\rho\,e^{p\lambda}, \rho_x\,e^{(p-1)\lambda})&=e^{-\lambda}G(\rho,\rho_x)\\
L(\rho\,e^{p\lambda}, \rho_x\,e^{(p-1)\lambda})&=e^{-\lambda}L(\rho,\rho_x).
\end{aligned} \tag{2.4b}$$

We shall only make use of the generic symmetries P_0, P_1 and W. Subalgebras leading to invariant solutions are the following.

1. $P_1 + aW$

$$\rho = \rho(t) \qquad \omega = ax + \phi(t) \tag{2.5}$$

2. $P_0 + aW$

$$\rho = \rho(x) \qquad \omega = at + \phi(x) \tag{2.6}$$

3. $p_0 + bP_1 + aW, b \neq 0$

$$\rho = \rho(\xi) \qquad \omega = at + \phi(\xi) \qquad \xi = x - bt. \tag{2.7}$$

The functions ρ and ϕ in each case satisfy coupled systems of ordinary differential equations.

Subalgebras that can lead to partially invariant solutions are

$$\{P_1, W\} \qquad \{P_0, W\} \qquad \{P_0 + bP_1, W\} \qquad b \neq 0. \tag{2.8}$$

3. The Subalgebra $\{P_1, W\}$

The solutions have the form

$$\rho = \rho(t) \qquad \omega = \omega(x, t) \qquad \omega_x \neq \text{const}. \tag{3.1}$$

Equation (2.1) can be rewritten as

$$\omega_t = -F + L\omega_x - \omega_x^2 \tag{3.2}$$

$$\omega_{xx} - G\omega_x = K - \frac{\rho_t}{\rho}. \tag{3.3}$$

Let us consider the case $G \neq 0$ and $G = 0$ separately.

(a) $G \neq 0$

Equation (3.3) is a linear inhomogeneous equation for ω with coefficient independent of x. Its general solution is

$$\omega = \beta(t)\, e^{xG} + \frac{1}{G}\left(\frac{\rho_t}{\rho} - K\right)x + \alpha(t). \tag{3.4}$$

Substituting (3.4) in (3.2) we obtain terms proportional to $e^{2xG}, e^{xG}, x\, e^{G}, x$ and independent of x. Equating coefficients of like terms, we find

$$\beta(t) = 0 \qquad \frac{1}{G}\left(\frac{\rho_t}{\rho} - K\right) = a = \text{const} \tag{3.5}$$

and we obtain an invariant solution of the form (2.5)

(b) $G = 0$

The solution of equation (3.3) is

$$\omega = \frac{1}{2}\left(K - \frac{\rho_t}{\rho}\right)x^2 + \alpha(t)x + \beta(t). \tag{3.6}$$

We substitute into (3.2) and set equal coefficients of x^2, x and 1. We obtain an equation for ρ, namely

$$\rho_t = \left[K(\rho, 0) - \frac{1}{2t}\right]\rho \tag{3.7}$$

and expressions for $\alpha(t)$ and $\beta(t)$:

$$\alpha(t) = \frac{\alpha_0}{t} + \frac{1}{2t}\int L(\rho, 0)\,\mathrm{d}t \tag{3.8}$$

$$\beta(t) = \int [-F(\rho, 0) + L(\rho, 0)\alpha - \alpha^2]\,\mathrm{d}t + \beta_0. \tag{3.9}$$

We thus have a solution

$$\rho = \rho(t, c_1) \tag{3.10a}$$

$$\omega = \frac{x^2}{4t} + \left[\alpha_0 + \tfrac{1}{2}\int L(\rho, 0)\,\mathrm{d}t\right]\frac{x}{t} + \beta(t) \tag{3.10b}$$

where $\rho(t, c_1)$ is a solution of equation (3.7), $\beta(t)$ is given by (3.9) and c_1, α_0 and β_0 are constants.

In the special case when $L(\rho, \rho_x)$ satisfies $L(\rho, 0) = 0$ equation (2.1) is Galilei invariant. The algebra $B + aW$ then leads to an invariant solution of the form

$$\rho = \rho(t) \qquad \omega = \frac{x^2}{4t} + a\frac{x}{t} + \phi(t). \tag{3.11}$$

We conclude that (3.10) represents a Galilei invariant solution for $L(\rho, 0) = 0$ and a genuinely partially invariant one for $L(\rho, 0) \neq 0$.

Now let us restrict to the particular case (1.8), studied by Clarkson [5]. We must have $G = 0$, hence $a_1 = 0$; moreover we have $K = 0$. Equations (3.6), ..., (3.10) yield the partially invariant solution

$$\begin{aligned} \rho &= \frac{\rho_0}{\sqrt{t}} \\ \omega &= \frac{x^2}{4t} + \left(\alpha_0 - \frac{a_2\rho_0^2}{2}\ln t\right)\frac{x}{t} + \mathrm{d}\rho_0^2 \ln t + (\alpha_0^2 - c\rho_0^4)\frac{1}{t} - \alpha_0 a_2 \rho_0^2 \frac{\ln t}{t} + \frac{a_2\rho_0^4}{4}\frac{\ln^2 t}{t} + \omega_0. \end{aligned} \tag{3.12}$$

The solution (3.12) does fit into the scheme of 'direct method reductions' with the ansatz of [5].

The partially invariant solution (3.10*a*) is obtained quite explicitly for any choice of the coefficients F, K and L in the NLSE (1.1), as long as we have $G(\rho, 0) = 0$.

4. The subalgebra $\{P_0, W\}$

4.1. The partially invariant solutions

The solutions have the form

$$\rho = \rho(x) \qquad \omega = \omega(x, t) \qquad \omega_t \neq \text{const.} \tag{4.1}$$

Equation (2.1*b*) can be solved for ω_x and we have

$$\omega_x = \alpha(t)A(x) + B(x) \tag{4.2}$$

where $A(x)$ and $B(x)$ satisfy

$$A_x = \left(-\frac{2\rho_x}{\rho} + G\right)A \tag{4.3}$$

$$B_x = \left(-\frac{2\rho_x}{\rho} + G\right)B + K + \frac{L\rho_x}{\rho}. \tag{4.4}$$

We then obtain ω_t from equation (2.1*a*) as

$$\omega_t = \left(\frac{\rho_{xx}}{\rho} - F - G\frac{\rho_x}{\rho}\right) + L(\alpha A + B) - \alpha^2 A^2 - 2\alpha AB - B^2. \tag{4.5}$$

Compatibility of (4.2) and (4.5) then implies

$$\dot{\alpha} = \tfrac{1}{2}\lambda\alpha^2 + \mu\alpha + \nu \qquad \lambda, \mu, \nu = \text{const} \qquad (\lambda \neq 0) \tag{4.6}$$

$$A = -\frac{\lambda}{4}x \qquad B = \frac{1}{4}\left[-\mu x + 2L - \frac{\Lambda_0}{x}\right] \qquad \Lambda_0 = \text{const} \tag{4.7}$$

and we obtain an expression for the phase

$$\omega(x, t) = -\frac{\alpha\lambda + \mu}{8}x^2 - \frac{\lambda\Lambda_0}{8}\int \alpha\, dt + ct - \frac{\Lambda_0}{4}\ln x + \frac{1}{2}\int L\, dx + \omega_0. \tag{4.8}$$

Integrating (4.6) we obtain more explicit expressions for the phase. We put

$$\Delta = \mu^2 - 2\lambda\nu \tag{4.9}$$

and obtain

1. $\Delta = 0$

$$\omega(x, t) = \frac{x^2}{4t} + \frac{\Lambda_0}{4}\ln\left(\frac{t}{x}\right) + \left(c + \frac{\Lambda_0\mu}{8}\right)t + \frac{1}{2}\int L\, dx + \omega_0 \tag{4.10a}$$

2. $\Delta > 0$

$$\omega(x, t) = \frac{\sqrt{\Delta}}{8}x^2\tanh\left(\frac{1}{2}\sqrt{\Delta}t\right) + \frac{\Lambda_0}{4}\ln\left[\frac{\cosh(\frac{1}{2}\sqrt{\Delta}t)}{x}\right] + \left(c + \frac{\Lambda_0\mu}{8}\right)t + \frac{1}{2}\int L\, dx + \omega_0 \tag{4.10b}$$

3. $\Delta < 0$

$$\omega(x, t) = -\frac{x^2}{8}\sqrt{|\Delta|}\tan\left(\frac{1}{2}\sqrt{|\Delta|}t\right) + \frac{\Lambda_0}{4}\ln\left[\frac{\cos(\frac{1}{2}\sqrt{|\Delta|}t)}{x}\right] + \left(c + \frac{\mu\Lambda_0}{8}\right)t + \frac{1}{2}\int L\, dx + \omega_0. \tag{4.10c}$$

We have dropped a constant when integrating equation (4.6) and hence t in (4.10) can be replaced by $t - t_0$ everywhere. For t_0 real this constant is recovered by a time translation, i.e. a symmetry transformation. However, in the case of equation (4.10*b*) we can also take $t_0 = -(i\pi/\sqrt{\Delta})$ and redefine the constant ω_0 to obtain a different expression for $\omega(x, t)$ in which the functions $\tanh(\sqrt{\Delta}t/2)$ and $\cosh(\sqrt{\Delta}t/2)$ are replaced by $\coth(\sqrt{\Delta}t/2)$ and $\sinh(\sqrt{\Delta}t/2)$, respectively.

By using the expressions given in the equation (4.7), equations (2.1) also imply that ρ is a solution of the equations

$$\rho_x = \frac{1}{2}\left(G - \frac{1}{x}\right)\rho \tag{4.11}$$

$$L_{\rho_x}\rho_{xx} + L_\rho \rho_x + \Lambda_0\left(G - \frac{2\rho_x}{\rho}\right)^2 - GL - 2K = 0 \tag{4.12}$$

$$\rho_{xx} - G\rho_x - F\rho + \frac{L^2}{4}\rho = \left\{\frac{\Delta}{16}\left(-\frac{2\rho_x}{\rho} + G\right)^{-2} + \frac{\Lambda_0^2}{16}\left(-\frac{2\rho_x}{\rho} + G\right)^2 + \frac{\mu\Lambda_0}{8} + c\right\}\rho. \tag{4.13}$$

Differentiating equation (4.11) with respect to x, we obtain an equation involving ρ_{xx}. Equations (4.11) ... (4.13) involve consistency conditions for the functions F, G, K and L, necessary for partially invariant solutions to exist.

4.2. Compatibility conditions on the coefficients in the NLSE

Two cases will be considered separately.

1. $2 - \rho G_{\rho_x} = 0$.

We then have

$$G = \frac{2\rho_x}{\rho} + H(\rho) \qquad H_\rho \neq 0 \tag{4.14}$$

and equation (4.11) reduces to a functional equation for ρ:

$$H(\rho) = \frac{1}{x} \qquad \rho = H^{-1}\left(\frac{1}{x}\right). \tag{4.15}$$

Equations (4.12) and (4.13) then determine K and F in terms of two free functions $H(\rho)$ and $L(\rho, \rho_x)$. We have

$$K = \frac{1}{2}\left\{L_{\rho_x}\left(\frac{H}{H_\rho}\right)^3(2H_\rho^2 - HH_{\rho\rho}) - L_\rho\frac{H^2}{H_\rho} + L\left(\frac{2H^2}{\rho H_\rho} - H\right) + \Lambda_0 H^2\right\} \tag{4.16}$$

$$F = \frac{1}{\rho^2}\left(\frac{H}{H_\rho}\right)^3(3\rho H_\rho^2 - \rho HH_{\rho\rho} - 2HH_\rho) + \frac{L^2}{4} - \frac{\Delta}{16H^2} - \frac{\Lambda_0^2}{16}H^2 - \frac{\mu\Lambda_0}{8} - c. \tag{4.17}$$

Since F, K and L depend on both ρ and ρ_x there is a certain freedom in equations (4.14), (4.16) and (4.17). A consequence of (4.15) is

$$\rho_x = -\frac{H^2}{H_\rho} \tag{4.18}$$

and we can use (4.18) to replace ρ_x in the expressions for G, K and L. Conversely, H_ρ can be replaced anywhere, using (4.18). Thus, we can replace (4.14) by

$$G = -\frac{2H^2}{\rho H_\rho} + H. \tag{4.19}$$

2. $2 - \rho G_{\rho_x} \neq 0$.

This time ρ must be determined from the first order ODE (4.11) and (4.12), (4.13) imply:

$$K=\frac{1}{2-\rho G_{\rho_x}}L_{\rho_x}\left[\frac{1}{2}G_\rho\rho\rho_x+\frac{\rho_x^2}{\rho}+\frac{1}{2}\rho\left(-\frac{2\rho_x}{\rho}+G\right)^2\right]+\frac{1}{2}L_\rho\rho_x+\frac{1}{2}\Lambda_0\left(G-\frac{2\rho_x}{\rho}\right)^2-\frac{1}{2}GL \tag{4.20}$$

$$F=\frac{1}{\rho(2-\rho G_{\rho_x})}\left[\rho\rho_xG_\rho+\rho G^2-4\rho_xG+6\frac{\rho_x^2}{\rho}\right]-\frac{G\rho_x}{\rho}+\frac{L^2}{4}-\frac{\Delta}{16}\left(-\frac{2\rho_x}{\rho}+G\right)^{-2}-\frac{\Lambda_0^2}{16}\left(-\frac{2\rho_x}{\rho}+G\right)^2+\frac{\mu\Lambda_0}{8}-c \tag{4.21}$$

where $G(\rho,\rho_x)$ and $L(\rho,\rho_x)$ are arbitrary ($G_{\rho_x}\neq 2\rho^{-1}$).

4.3. Special cases

We shall now extract several particularly simple special cases from the general formula.

We first consider the case $2-\rho G_{\rho_x}=0$.

1. $L=L_0$, $H=H_0\rho^n$, $n\neq 0$.

Equations (4.16), (4.17) and (4.19) in this case imply

$$G=\frac{2\rho_x}{\rho}+H_0\rho^n \qquad K=K_0\rho^{2n}+K_1\rho^n \qquad F=F_1\rho^{2n}+F_2\rho^{-2n}+F_0 \tag{4.22}$$

where K_0, F_1, F_2 and F_0 are constants. The solution has modulus

$$\rho=\left(\frac{1}{H_0}\right)^{1/n}x^{-1/n}. \tag{4.23}$$

The phase of the solution satisfies (4.8) (and hence (4.10)).

The constants $H_0(\neq 0)$, L_0, K_0, F_0 and F_2 are arbitrary, but we must have

$$F_1=\frac{2n-1}{n^2}H_0^2-\frac{K_0^2}{4H_0^2} \qquad K_1=\frac{2-n}{2n}L_0H_0.$$

The constants λ, μ, ν, Λ_0 and c, figuring in the phase $\omega(x,t)$ satisfy

$$c=-F_0-\tfrac{1}{8}\Lambda_0\mu+\frac{L_0^2}{4} \qquad \Lambda_0=\frac{2K_0}{H_0^2} \qquad \Delta=-16H_0^2F_2. \tag{4.25}$$

We see that G and K are polynomials for n positive integer, however F is not, unless we set $F_2=0$. In this particular case equation (1.1) has an additional symmetry, namely dilations combined with a time dependent change of phase. The generator of these transformations is

$$D=2t\partial_t+x\partial_x-\frac{1}{n}\rho\partial_\rho-2F_0t\partial_\omega. \tag{4.26}$$

In this particular case ($F_2=0$) the obtained partially invariant solution is actually invariant under the group generated by $D+aW$.

For $F_2 \neq 0$ equation (1.1) is invariant only under the group generated by $\{P_0, P_1, W\}$ of (2.2). Hence in this case we have a genuinely partially invariant solution.

2. $K=0$, $G=(2\rho_x/\rho)+(\rho^2/\rho_0^2)$.

In this case equations (4.19), (4.16) and (4.17) imply

$$H=\frac{\rho^2}{\rho_0^2} \qquad L=L_0\rho^2+l(\xi) \qquad \xi=\rho_x+\frac{1}{2}\frac{\rho^3}{\rho_0^2}$$
$$F=F_1\rho^4+F_2\rho^{-4}+\tfrac{1}{2}L_0 l\rho^2+F_0+\tfrac{1}{4}l^2 \tag{4.27}$$

where $\rho_0 \neq 0$, L_0, F_0, F_1 and F_2 are constants and $l(\xi)$ is an arbitrary function.

The solution is given by

$$\rho=\frac{\rho_0}{\sqrt{x}} \tag{4.28}$$

and equation (4.8), respectively.

The constants ρ_0, L_0, F_0 and F_2 are free but F_1 must satisfy

$$F_1=\frac{3}{16\rho_0^4}(4+\rho_0^4L_0^2). \tag{4.29}$$

The constants involved in the phase $\omega(x, t)$ satisfy

$$\Lambda_0=\rho_0^2L_0 \qquad \Delta=\frac{-16F_2}{\rho_0^4}$$
$$c=-F_0-\frac{\mu}{8}\rho_0^2L_0. \tag{4.30}$$

If we set $F_2=0$ and $l(\xi)=0$, our solution reduces to an invariant one. In all other cases it is a genuinely partially invariant solution.

Now let us consider an example of the case when we have $2-\rho G_{\rho_x} \neq 0$.

3. $$G=B\left(\frac{2}{A\rho}+1\right)e^{-A\rho}(A, B=\text{const}, AB\neq 0) \qquad L=L_0. \tag{4.31}$$

From (4.11) we obtain

$$\rho=\frac{1}{A}\ln Bx \tag{4.32}$$

so that we can express ρ_x in terms of ρ. The expressions (4.20) and (4.21) simplify to

$$K=K_0\,e^{-2A\rho}-\tfrac{1}{2}L_0B\left(\frac{2}{A\rho}+1\right)e^{-A\rho}$$
$$F=-\left(\frac{2B^2}{\rho^2A^2}+\frac{2B^2}{\rho A}+\frac{K_0^2}{4B^2}\right)e^{-2A\rho}+F_2\,e^{2A\rho}+F_0. \tag{4.33}$$

The constants in the phase (4.8) satisfy

$$\Lambda_0=\frac{2K_0}{B^2} \qquad c=-\frac{\mu K_0}{4B^2}-F_0+\frac{L_0^2}{4} \qquad \Delta=-16B^2F_2. \tag{4.34}$$

We mention that NLSEs with exponential nonlinearities in the coefficients do occur in physical contexts [15].

5. The subalgebra $\{P_0+aP_1, W\}$

The subalgebras $\{P_0+aP_1, W\}$ for $a\neq 0$ is conjugated to $\{P_0, W\}$ by a Galilei transformation. Hence it is of interest to consider this subalgebra only if the equation (1.1) is not Galilei invariant, i.e. if we have $GL\neq 0$.

Even in this case the results are very similar to those of section 4. A partially invariant solution will in this case have the form

$$\rho=\rho(\xi) \qquad \omega=\omega(\xi, t) \qquad \xi=x-at. \tag{5.1}$$

The reduced system is

$$\omega_t=-\omega_\xi^2+(L+a)\omega_\xi+\left[\frac{\rho_{\xi\xi}}{\rho}-F-G\frac{\rho_\xi}{\rho}\right] \tag{5.2a}$$

$$\rho\omega_{\xi\xi}+(2\rho_\xi-G\rho)\omega_\xi=K\rho+(L+a)\omega_\xi. \tag{5.2b}$$

The system (5.2) differs from the corresponding one for the subalgebra $\{P_0, W\}$ only by the fact that L is replaced by $L+a$ and x by ξ. This is the only modification in the entire analysis.

Thus $\rho(\xi)$ is obtained by solving the equation

$$\rho_\xi=\frac{1}{2}\left(G-\frac{1}{\xi}\right)\rho \tag{5.3}$$

and $\omega(x, t)$ is given by (4.10) with

$$x\to\xi=x-at \qquad L\to L+a. \tag{5.4}$$

Compatibility must be considered for $2-\rho G_{\rho\xi}=0$ and $2-\rho G_{\rho\xi}\neq 0$ separately and the results of section 4 pertain with the appropriate replacements (5.4).

The particular examples of subsection 4.3 also carry over and in this case we obtain travelling wave solutions.

6. Conclusions

We have shown that partially invariant solutions, when they exist, are quite easy to obtain for a very general class of NLSEs (1.1). The existence of such solutions imposes constraints on the functions F, G, K and L. The form of these constraints depends on the chosen subgroup of the symmetry group. Thus, for the subgroup $\{P_1, W\}$ we have obtained the condition $G=0$. For $\{P_0, W\}$ we have (4.14), (4.16) and (4.17), or (4.20) and (4.21).

It is instructive to compare partially invariant solutions with solutions obtained by the 'non-classical method' of Bluman and Cole [18], interpreted in terms of 'conditional symmetries' [19].

The symmetry algebra of the system (2.1) is realized by vector fields of the form

$$\hat{v}=\xi\partial_x+\tau\partial_t+\phi_1\partial_\rho+\phi_2\partial_\omega \tag{6.1}$$

where ξ, τ, ϕ_1 and ϕ_2 are functions of x, t, ρ and ω, to be determined from the condition that the second prolongation [3, 4] $pr^{(2)}\hat{v}$ should annihilate the equations (2.1) on their solution set. Conditional symmetries [19] annihilate the equations (2.1) only on a subset of solutions, namely those that also satisfy the conditions

$$\xi\rho_x+\tau\rho_t-\phi_1=0 \tag{6.2a}$$

$$\xi\omega_x+\tau\omega_t-\phi_2=0. \tag{6.2b}$$

A generalization of partially invariant solutions would be obtained by allowing $\omega(x, t)$ to depend on both x and t, but requesting that ρ be a function of one variable ξ, obtained from a conditional symmetry. An investigation of this possibility is in progress.

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